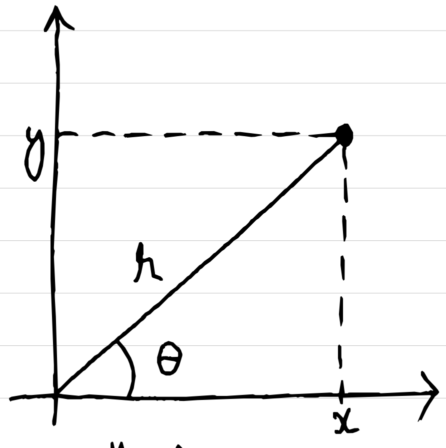
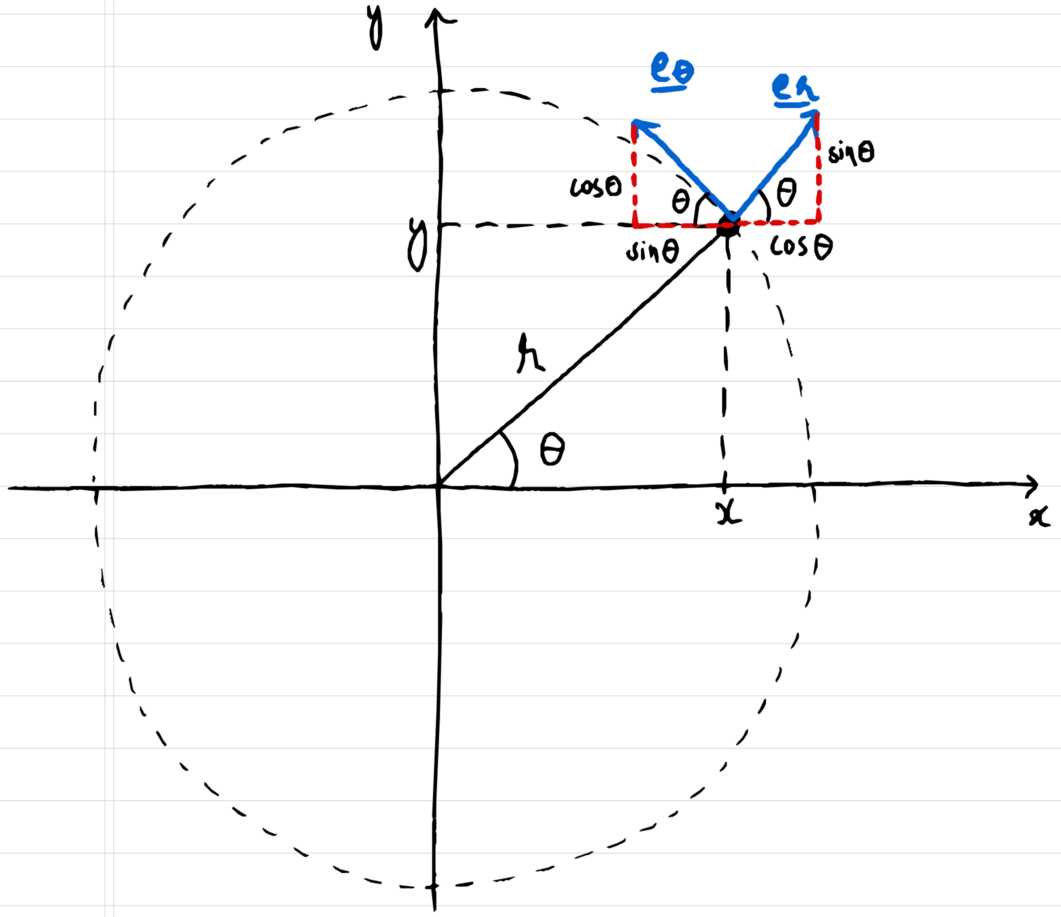
5) Polar Coordinates

5.1) Basics of Polar Coordinates

Sometimes it is more convenient to use polar coordinates (r, θ) rather than Cartesian coordinates (x, y)





The relation between polar and Cartesian coordinates is given by

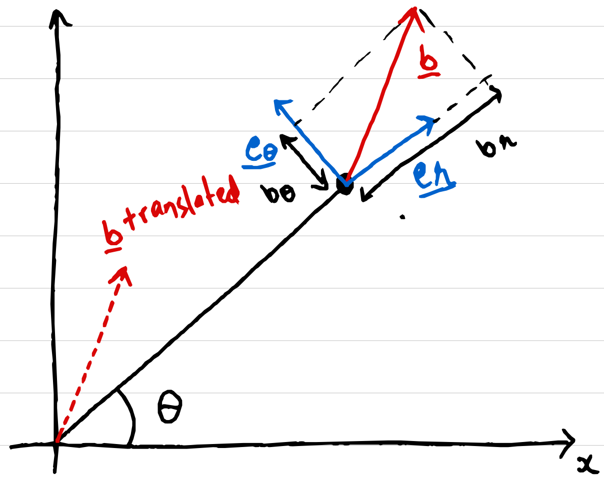
$$\boxed{x = r\cos\theta \qquad y = r\sin\theta}$$

→ Just like $\underline{i}$ and $\underline{j}$, $\underline{e_\theta}$ and $\underline{e_r}$ form a **basis** of the 2D plane

→ Just like $\{\underline{i}, \underline{j}\}$, $\{\underline{e_\theta}, \underline{e_r}\}$ form an orthonormal basis

At any position $\underline{x}$ on the xy plane we can introduce two unit vectors $\underline{e}_r$ and $\underline{e}_\theta$ (unit vectors in radial and azimuthal directions) as shown in Fig on page 1.

Any vector associated with point $\underline{x}$ (eg the velocity of particle $\dot{\underline{x}}(t)$ whose position at time t is $\underline{x}(t)$) can be presented in the form



Any vector $b \in \mathbb{R}^2$ can be expressed as a linear combination of $\underline{e}_\theta$ and $\underline{e}_r$

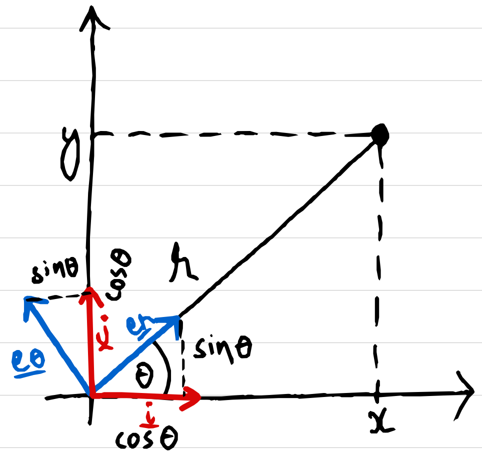
$$\boxed{\underline{b} = b_r \underline{e}_r + b_\theta \underline{e}_\theta}$$

Scalars: b_r is the radial component

b_θ is the azimuthal component

of vector $\underline{b}$

Unit vectors $\underline{e_r}$ and $\underline{e_\theta}$ can be expressed in terms of cartesian basis vectors $\underline{i}$ and $\underline{j}$.



$\underline{e_r} = 1.\cos\theta\,\underline{i} + 1.\sin\theta\,\underline{j}$

$\Rightarrow$

$$\boxed{\underline{e_r} = \cos\theta\,\underline{i} + \sin\theta\,\underline{j}}$$

$\underline{e_\theta} = \pm\sin\theta\,\underline{i} \mp \cos\theta\,\underline{j}$ (since $\underline{e_\theta}$ is perpendicular to $\underline{e_r}$)

$\Rightarrow$

$$\boxed{\underline{e_\theta} = \sin\theta\,\underline{i} - \cos\theta\,\underline{j}}$$ (from diagram)

Note:

$\begin{pmatrix} x \\ y \end{pmatrix}$ is perpendicular to $\lambda\begin{pmatrix} -y \\ x \end{pmatrix}$ or $\lambda\begin{pmatrix} y \\ -x \end{pmatrix}$

$\lambda \neq 0$

So

$$\boxed{\begin{aligned} \underline{e_r} &= \cos\theta\,\underline{i} + \sin\theta\,\underline{j} \\ \underline{e_\theta} &= \sin\theta\,\underline{i} - \cos\theta\,\underline{j} \end{aligned}}$$

Assume that polar angle θ changes with time i.e. polar angle is a function of time $\theta(t)$

Computing $\dot{\underline{e}}_r$ and $\dot{e}_\theta$

$$\dot{\underline{e}}_r = \frac{d}{dt}\left(\cos(\theta)\,\underline{i} + \sin\theta\,\underline{j}\right)$$

$$= -\sin(\theta)\,\dot{\theta}\,\underline{i} + \cos(\theta)\,\dot{\theta}\,\underline{j}$$

$$= \dot{\theta}\left(-\sin(\theta)\,\underline{i} + \cos(\theta)\,\underline{j}\right)$$

$$= \dot{\theta}\,\underline{e}_\theta$$

$$\Rightarrow \boxed{\dot{\underline{e}}_r = \dot{\theta}\,\underline{e}_\theta = -\sin(\theta)\,\dot{\theta}\,\underline{i} + \cos(\theta)\,\dot{\theta}\,\underline{j}}$$

Similarly

$$\dot{\underline{e}}_\theta = \frac{d}{dt}\left(-\sin(\theta)\,\underline{i} + \cos(\theta)\,\underline{j}\right)$$

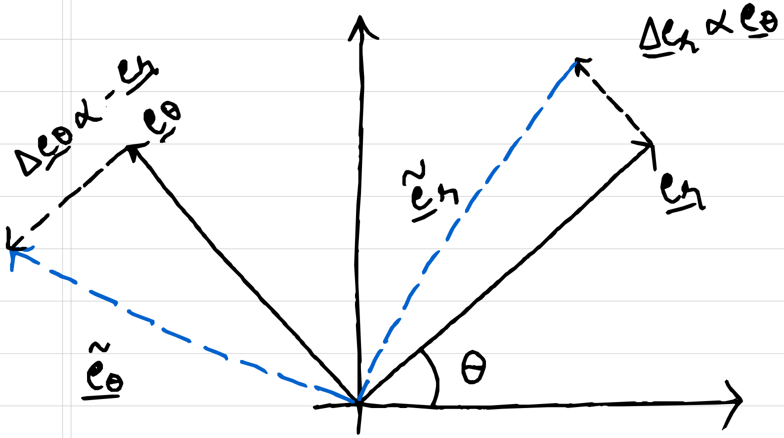
$$= -\cos(\theta)\,\dot{\theta}\,\underline{i} - \sin(\theta)\,\dot{\theta}\,\underline{j}$$

$$= -\dot{\theta}\left(\cos(\theta)\,\underline{i} + \sin(\theta)\,\underline{j}\right)$$

$$= -\dot{\theta}\,\underline{e}_r$$

$$\Rightarrow \boxed{\dot{\underline{e}}_r = -\dot{\theta}\,\underline{e}_r = -\cos(\theta)\,\dot{\theta}\,\underline{i} - \sin(\theta)\,\dot{\theta}\,\underline{j}}$$

(! Using $\frac{d\underline{i}}{dt} = \frac{d\underline{j}}{dt} = 0$, i.e. they are constant)



As we can see from the above diagram and previous eqn

$\Delta \underline{e_\theta} \propto -\underline{e_r}$: Change in $\underline{e_\theta}$ is proportional to the opposite direction of $\underline{e_r}$

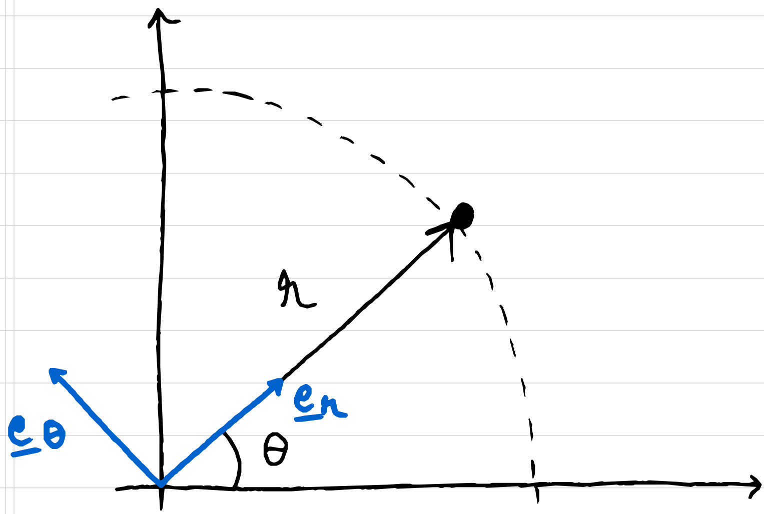
$\Delta \underline{e_r} \propto \underline{e_\theta}$: Change in $\underline{e_r}$ is proportional to $\underline{e_\theta}$

5.2) Kinematics in Polar Coordinates

Defn: Position vector in polar coordinates:

In polar coordinates, the position vector of a particle is simply

$$\boxed{\underline{x} = r \cdot \underline{e_r}}$$



Computing velocity vector in polar coordinates

$$\underline{v}(t) = \dot{\underline{x}}(t)$$

$$\underline{v}(t) = \dot{\underline{x}}(t) = \frac{d}{dt}\left(r \cdot \underline{e_r}\right) \qquad \text{(apply product rule)}$$

$$= \dot{r}\underline{e_r} + r\dot{\theta}\underline{e_\theta}$$

linear/radial velocity

rotational velocity

Defn: Velocity vector in polar coordinates

$$\boxed{\underline{v}(t) = \underline{\dot{x}}(t) = \dot{r}\,\underline{e_r} + r\dot{\theta}\,\underline{e_\theta}}$$

Computing acceleration vector $\underline{a}(t)$

$$\underline{a}(t) = \underline{\ddot{x}} = \frac{d}{dt}\left(\underline{\dot{x}}(t)\right)$$

$$= \frac{d}{dt}\left(\dot{r}\,\underline{e_r} + r\dot{\theta}\,\underline{e_\theta}\right) \quad \text{(Apply product rule)}$$

$$= \ddot{r}\,\underline{e_r} + \dot{r}\,\underline{\dot{e}_r} + \dot{r}\dot{\theta}\,\underline{e_\theta} + r\ddot{\theta}\,\underline{e_\theta} + r\dot{\theta}\,\underline{\dot{e}_\theta}$$

$$= \ddot{r}\,\underline{e_r} + \dot{r}\dot{\theta}\,\underline{e_\theta} + \dot{r}\dot{\theta}\,\underline{e_\theta} + r\ddot{\theta}\,\underline{e_\theta} - r\dot{\theta}^2\,\underline{e_r}$$

$$= (\ddot{r} - r\dot{\theta}^2)\,\underline{e_r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\underline{e_\theta}$$

$$\Rightarrow \underline{a}(t) = \underline{\ddot{x}}(t) = (\underbrace{\ddot{r}}_{\text{motion with no rotation}} \underbrace{- r\dot{\theta}^2}_{\text{centripetal acceleration}})\,\underline{e_r} + (\underbrace{2\dot{r}\dot{\theta}}_{\text{coriolis effect}} + r\ddot{\theta})\,\underline{e_\theta}$$

Defn: Acceleration vector in polar coordinates

$$\boxed{\underline{a}(t) = \underline{\ddot{x}}(t) = (\ddot{r} - r\dot{\theta}^2)\,\underline{e_r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\underline{e_\theta}}$$

Defn: Centripetal acceleration

In $\underline{a}(t) = \underline{\ddot{x}}(t) = (\ddot{r} - \underbrace{r\dot{\theta}^2}_{1})\underline{e_r} + (r\ddot{\theta} + \underbrace{2\dot{r}\dot{\theta}}_{2})\underline{e_\theta}$
the term

$$\boxed{-r\dot{\theta}^2}$$

is the centripetal acceleration

Note: $r > 0$ and $\dot{\theta}^2 > 0$

$\Rightarrow -r\dot{\theta}^2 < 0$, i.e. it is negative.

So opposite to direction of $\underline{e_r}$, i.e. centripetal acceleration is towards origin

Centripetal acceleration is present for instance when particle is moving in a circle.

The second additional term,

$$\boxed{2\dot{r}\dot{\theta}}$$

is the coriolis effect.

↳ explains why eg a ball thrown from a merry go round seems to curve.

Note: $2\dot{r}\dot{\theta}\underline{e_\theta}$ is non-zero only if both $\dot{r}$ and $\dot{\theta}$ are non-zero.

i.e. $2\dot{r}\dot{\theta}\underline{e_\theta} \neq 0 \Rightarrow \dot{r} \neq 0$ and $\dot{\theta} \neq 0$

i.e. for Coriolis effect r and θ both must change

Defn: Equations of motion in polar Coordinates

$$\underline{F} = m\ddot{\underline{x}} = m\begin{pmatrix} \ddot{r} - r\dot{\theta}^2 \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} \end{pmatrix} = \begin{pmatrix} F_r \\ F_\theta \end{pmatrix}$$

$$\Rightarrow \boxed{m\begin{pmatrix} \ddot{r} - r\dot{\theta}^2 \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} \end{pmatrix} = \begin{pmatrix} F_r \\ F_\theta \end{pmatrix}}$$

Another way of writing

$$\underline{F} = m\ddot{x} = m\left((\ddot{r} - r\dot{\theta}^2)\underline{e_r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\underline{e_\theta}\right)$$

$$= F_r \underline{e_r} + F_\theta \underline{e_\theta}$$

$$\Rightarrow$$

$$\boxed{\begin{cases} m(\ddot{r} - r^2\dot{\theta}^2) = F_r \\ m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = F_\theta \end{cases}}$$

- F_r is the radial component of force in direction e_r

$$\boxed{F_r = m(\ddot{r} - r\dot{\theta}^2)}$$

- F_θ is azimuthal component of force in direction e_θ

$$\boxed{F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})}$$

5.3) Circular Motion

In circular motion; r is constant, i.e.

$$\dot{r} = 0$$

i.e particle lies on circle with fixed radius from origin

So velocity vector gets reduced to

$$\underline{v}(t) = \underline{\dot{x}}(t) = \cancel{\dot{r}\underline{e_r}} + r\dot{\theta}\underline{e_\theta}$$

$$\Rightarrow \underline{v}(t) = r\dot{\theta}\underline{e_\theta}$$

Defn: Velocity vector in circular motion

In circular motion velocity is

$$\boxed{\underline{v}(t) = \underline{\dot{x}}(t) = r\dot{\theta}\underline{e_\theta}}$$

The acceleration vector reduces to

$$\underline{a}(t) = \underline{\ddot{x}}(t) = (\cancel{\ddot{r}} - r\dot{\theta}^2)\underline{e_r} + (\cancel{2\dot{r}}\dot{\theta} + r\ddot{\theta})\underline{e_\theta}$$

$$\Rightarrow \underline{a}(t) = -r\dot{\theta}^2\underline{e_r} + r\ddot{\theta}\underline{e_\theta}$$

Defn: Acceleration vector in circular motion

In circular motion, velocity is

$$\boxed{\underline{a}(t) = \underline{\ddot{x}}(t) = -r\dot{\theta}^2\underline{e_r} + r\ddot{\theta}\underline{e_\theta}}$$

5.3.1 Constant Circular Motion:

If $\dot{\theta} = w = const \qquad (\dot{\theta}(t) = w)$

let

$$\dot{\underline{x}} = \underline{v} = rw\underline{e}_\theta = \dot{x}(t)\underline{i} + \dot{y}(t)\underline{j}$$

$$\Rightarrow \underline{\dot{x}}(t) = rw(-\sin\theta\underline{i} + \cos\theta\underline{j}) = \dot{x}\underline{i} + \dot{y}\underline{j}$$

$$\Rightarrow -rw\sin\theta\underline{i} + rw\cos\theta\underline{j} = \dot{x}\underline{i} + \dot{y}\underline{j}$$

Therefore we get

$$\boxed{\begin{aligned}\dot{x}(t) &= -rw\sin(\theta)\\ \dot{y}(t) &= rw\cos(\theta)\end{aligned}}$$

These are <u>Cartesian components</u>

$$\dot{\theta}(w) = w \Rightarrow \theta(t) = wt + \theta_0$$

$$\boxed{\begin{aligned}\dot{x}(t) &= -rw\sin(wt + \theta_0)\\ \dot{y}(t) &= rw\cos(wt + \theta_0)\end{aligned}}$$

$$\downarrow$$

$$\boxed{\begin{aligned}x(t) &= r\cos(wt) + \theta_0\\ y(t) &= r\sin(wt) + \theta_0\end{aligned}}$$

↳ circular motion solution

5.4) Planets and Pendula

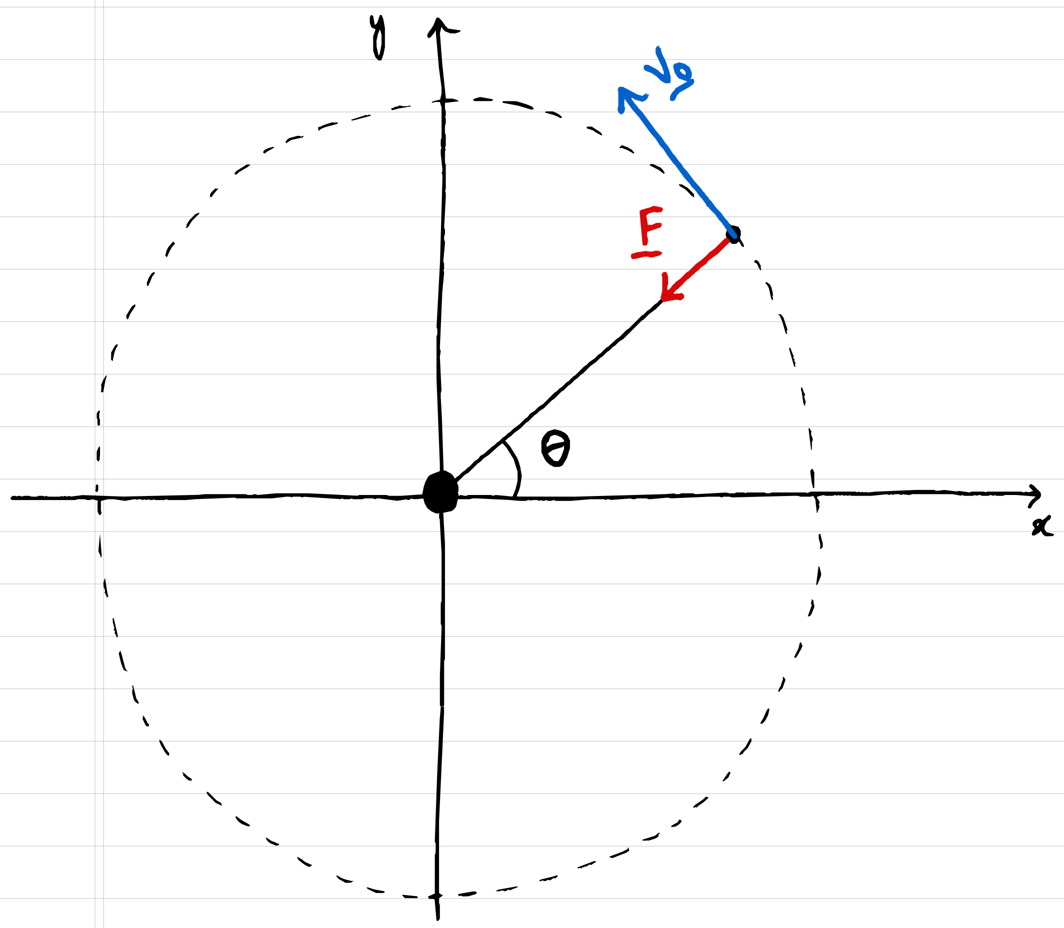
Example problem 1: (Circular motion):

Consider a planet of mass m which is moving with constant speed v_0 along a circular orbit. Let the radius of the orbit be R.

What is azimuthal velocity v_θ?

Solution:

Let the centre of the star be the origin of polar coordinates (r, θ)



The only force affecting affecting the planet is given by

$$\underline{F} = -\frac{GmM}{r^2} \underline{e_r}$$

So the eqn of motion becomes

$$m(\ddot{r} - r\dot{\theta}^2) = F_r = -\frac{GmM}{r^2} \qquad (*1)$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = F_\theta = 0 \qquad (*2)$$

Moving with constant speed v_0.

Constant radius $r(t) = R \Rightarrow \dot{r} = 0$ and $\ddot{r} = 0$

So: we get from $(*1)$

$$\cancel{m}(\cancel{\ddot{r}} - r\dot{\theta}^2) = -\frac{GM\cancel{m}}{R^2}$$

$$\Rightarrow -R\dot{\theta}^2 = -\frac{GM}{R^2} \qquad (*3)$$

From $(*2)$ we get

$$mR\ddot{\theta} = 0 \Rightarrow \ddot{\theta} = 0 \qquad (*4)$$

Solving (*4)

$$\ddot{\theta} = 0 \Rightarrow \theta = \omega t + \theta_0$$

(using initial conditions $\theta(0) = \theta_0$, $\dot{\theta}(0) = \omega$)

$$\Rightarrow \boxed{\theta(t) = \omega t + \theta_0}$$

Solving (*3)

$$\dot{\theta}^2 = \frac{GMm}{R^3} \Rightarrow \dot{\theta}(t) = \sqrt{\frac{GM}{R^3}} = \omega$$

$$\Rightarrow \dot{\theta}(t) = \sqrt{\frac{GM}{R^3}} = \omega$$

↳ constant

$$\dot{\theta}(t) = \sqrt{\frac{GM}{R^3}}$$

$$\underline{v}_0 = r\dot{\theta}\,\underline{e}_\theta \Rightarrow v_0 = |\underline{v}_0|$$

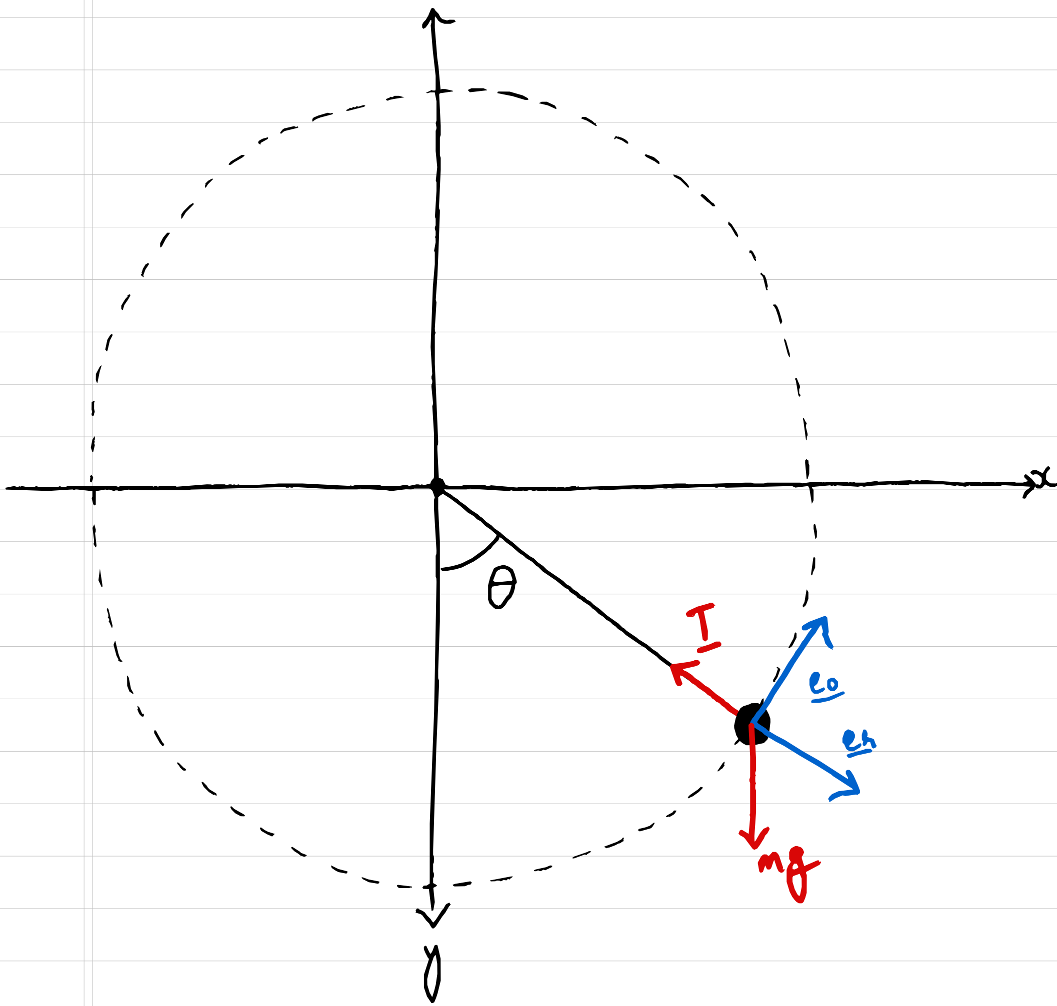
$$\Rightarrow v_0 = |R\dot{\theta}|\,|\underline{e}_\theta|$$

$$\Rightarrow v_0 = R.\sqrt{\frac{GM}{R^3}}$$

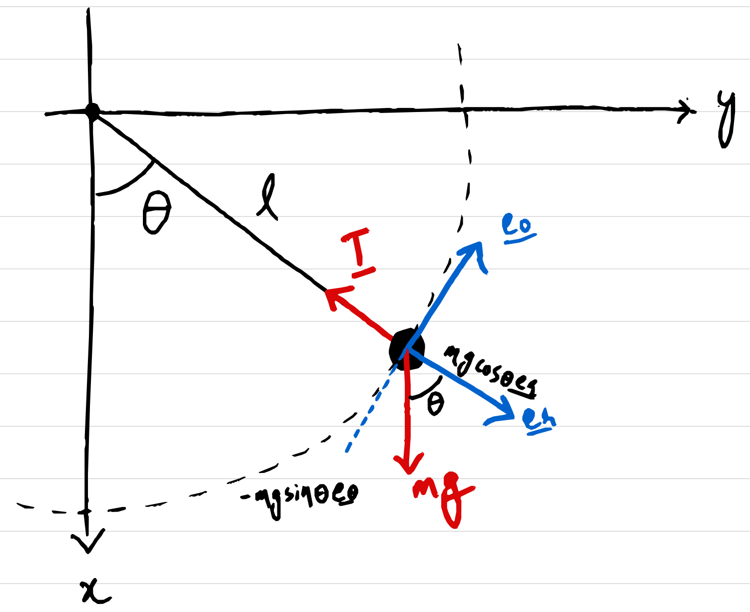
$$\Rightarrow v_0 = \sqrt{\frac{GM}{R}}$$

Example problem 2: (Simple pendulum):

Consider the motion of an ideal pendulum shown in figure below.



Making a zoomed in diagram:



Therefore from the following observations

- $\underline{T} = -T\underline{e_r}$
- $\underline{F} = m\underline{g} = m(mg\cos\theta\,\underline{e_r} - mg\sin\theta\,\underline{e_\theta})$
- Length is constant $\Rightarrow r(t) = \ell$
  $\Rightarrow \dot{r} = \ddot{r} = 0$

So total force on body is

$$\underline{F} = \underline{T} - m\underline{g}$$

$$\Rightarrow m\begin{pmatrix} \ddot{r} - r\dot{\theta}^2 \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} \end{pmatrix} = \begin{pmatrix} mg\cos\theta - T \\ -mg\sin\theta \end{pmatrix}$$

Imposing circular motion: length is constant.

$$r(t) \equiv \ell \Rightarrow \dot{r} \equiv 0 \Rightarrow \ddot{r} = 0$$

We get the following eqns of motion

$$-ml\dot{\theta}^2 = mg\cos\theta - T \qquad (*1)$$

$$ml\ddot{\theta} = -mg\sin\theta \qquad (*2)$$

$(*1)$ First of these allow us to determine T when $\theta(t)$ is known.

From $(*1)$

$$-m\ell\dot{\theta}^2 = -T + mg\cos\theta \Rightarrow \boxed{T = mg\cos\theta - m\ell\dot{\theta}^2}$$

$(*2)$ The second serves as an effective eqn of motion in azimuthal coordinate θ.
It is convenient to rewrite it as

$$\boxed{\ddot{\theta} = -\frac{g}{\ell}\sin\theta} \qquad (*3)$$

↳ exact differential eqn for $\underline{\theta(t)}$

Remark:

Note that eqn

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta$$

can be treated as one dimensional motion of particle of unit mass $m=1$ in the potential

$$\boxed{V(\theta) = -\frac{g}{\ell}\cos\theta}$$

so that we can write down the "energy"

$$\boxed{\tilde{E} = \frac{\dot{\theta}^2}{2} + V(\theta)}$$

and analyze motion qualitatively like in 3.7

- Evidently $\theta = 0$ is a constant soln of (*3).
In other words, $\theta = 0$ is an equilibrium position of the pendulum.

- Looking at motion near pendulum:

Assuming small oscillations, i.e. $|\theta| \ll 1$ i.e. θ is small,

By the fundemental theorem of engineering

$$\sin\theta \approx \theta$$

(Note: exam tip: small oscillations imply simple harmonic motion about stable equilibria)

As a result we obtain the following ODE (approximate ODE):

$$\ddot{\theta} = -\frac{g}{l}\theta$$

Upto the notation this is the same as the eqn of a simple harmonic oscillator. It describes small oscillations of the pendulum with angular frequency $\omega = \sqrt{g/l}$ and period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$$

Example problem 3: (planetary motion)

Consider a planet of mass m moving around a fixed star of mass M.

Let centre of star be the origin of polar coordinates (r, θ)
The only force acting on the planet is Newtownian gravitational force:

$$\underline{F} = -\frac{GmM}{r^2}\underline{e_r}$$

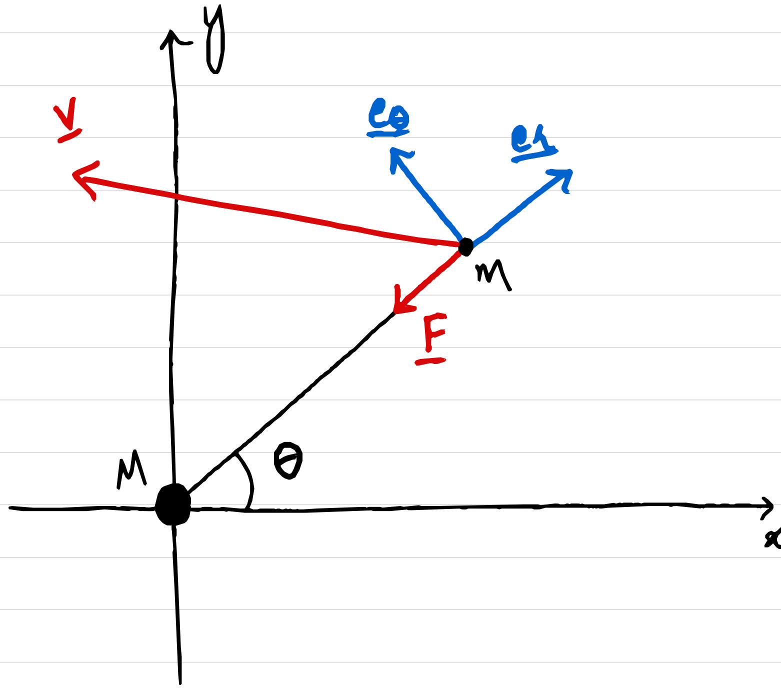
Equation of motion becomes

$$m(\ddot{r} - r\dot{\theta}^2) = -\frac{GMm}{r^2} \qquad (*1)$$

and

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \qquad (*2)$$

Diagram is drawn on next page



Dividing (*2) by m and multiplying by r, we find that

$$\ddot{r} + 2r\dot{r}\dot{\theta} = 0 \implies \frac{d}{dt}\left(r^2\dot{\theta}\right) = 0$$

Let $L = r^2\dot{\theta}$

This means that $L = r^2\theta$ is a constant of motion, since

$$\boxed{\dot{L} = \frac{d}{dt}\left(r^2\dot{\theta}\right) = 0}$$

Therefore

$$\boxed{L(t) = L(0)}$$

Defn: Angular momentum:

$\mathcal{L} = mL = mr^2\dot{\theta}$ is called <u>angular momentum</u>.
It is conserved in above example.

$$\boxed{\mathcal{L} = mL = mr^2\dot{\theta}}$$

We can use conservation of L to simply example problem 3.

Since L is a constant we have

$$\dot{\theta}(t) = \frac{L}{r^2(t)}$$

Substituting into first eqn (*1) and dividing by m, we get

$$\ddot{r} = \frac{-\gamma}{r^2} + \frac{L^2}{r^3} \qquad (*3)$$

where $GM = \gamma$

Eqn (*3)

$$\ddot{r} = \frac{-\gamma}{r^2} + \frac{L^2}{r^3} \qquad (*4)$$

is called the equation of radial motion and describes one-dimensional motion in radial direction. It can be solved (subject to appropriate initial conditions).

↳ This will then gives us $r(t)$

Then $r(t)$ is substituted into $\dot{\theta}(t)$ and by integration we find $\theta(t)$ such that

$$\theta(t) = \theta(0) + L\int_0^t \frac{ds}{r^2(s)} \qquad (*5)$$

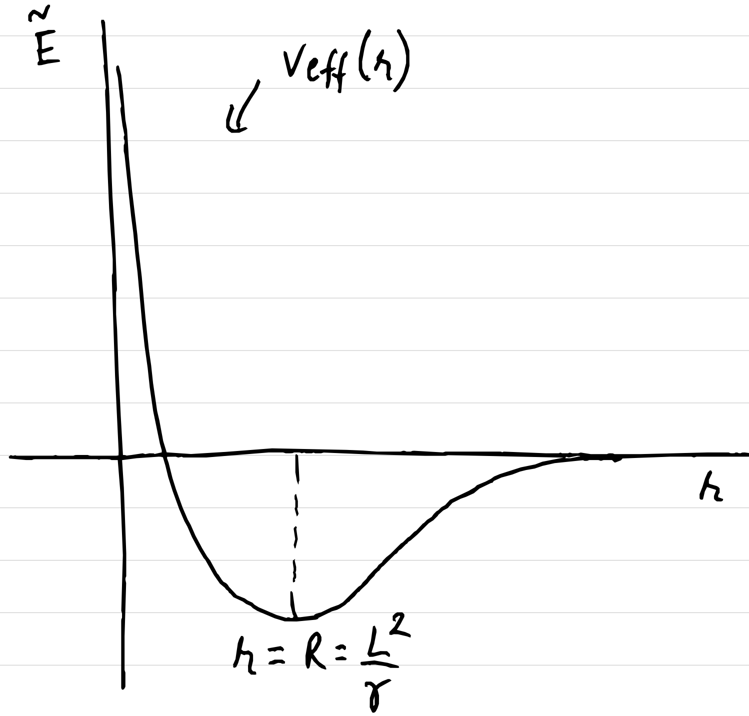
Thus (*4) and (*5) we can find potential. Since this is 1D, force is conservative

From (*4)

$$F_r = -dV_{eff}(r)$$

$$\Rightarrow V_{eff}(r) = -m\int\left(\frac{-\gamma}{s^2} + \frac{L}{s^3}\right)ds$$

$$= -\frac{m\gamma}{r} + \frac{mL^2}{2r^2} + C$$



Indeed

$$V'_{eff}(r) = \frac{m\gamma}{r^2} - \frac{mL^2}{r^3}$$

So we can we see from (*4)

$$\ddot{r} = -V'_{eff}(r)$$

The energy of the particle moving $V_{eff}(x)$ is

$$\tilde{E} = \frac{m\dot{r}^2}{2} + V_{eff}(r)$$

Now we can use what we already know about motion in a potential in one dimension

The sketch of $V_{eff}(r)$ is shown in Figure on previous page

The potential has a minimum point at

$$r = R = \frac{L^2}{\gamma}$$

$$V_{eff}(R) = \frac{L^2}{2R^2} - \frac{\gamma}{R} = \frac{-\gamma^2}{2L^2}$$

Note: This equilibrium point of radial motion is not a <u>true equilibrium</u>: It corresponds to a circular motion orbit of radius R such that azimuthal velocity is constant and equal to

$$R\dot{\theta} = \frac{L}{R} = \frac{\gamma}{L}$$

It follows that from the sketch of $V_{eff}'(r)$ that the motion of the particle is finite, i.e. takes place in a bounded region of $\mathbb{R}^2$, if $\tilde{E} < 0$

- $\tilde{E} < 0$, bound motion, planets orbit is ellipse
- $\tilde{E} \geq 0$: planets motion escapes to ∞
  - ↳ • $\tilde{E} = 0 \Rightarrow$ hyperbola
  - $\tilde{E} > 0 \Rightarrow$ parabola